**B. Math. Hons. II Year**
**Semesteral Exam 2001-2002**
**Algebra IV**

**Date: 08-05-2002** **Time: 9.45-12.45** **Instructor: B.Sury**

Answer all questions.

1. Let $G$ be a finite subgroup of $GL_n(\mathbb{C})$. Prove that $\exists\ A \in GL_n(\mathbb{C})$ such that $AGA^{-1} \subseteq U(n)$.

2. Let $G$ be a finite abelian group and $\hat{G}$ denote the group $Hom(G, \mathbb{C}^x)$. For any element $\alpha = \sum_g a_g \cdot g$ in $\mathbb{C}[G]$, compute the matrix of $\alpha$ with respect to the two bases $[g : g \in G]$ and $[\sum_g \chi(g)g : \chi \in \hat{G}]$ and conclude that $det(a_{gh^{-1}}) = \prod_{\chi \in \hat{G}} \sum_{g \in G} \chi(g)a_g$. What is this identity for $g = \mathbb{Z}/2$?

3. Let $G$ be a finite group and $f, g : G \to \mathbb{C}$ be class functions. Prove Plancherel's formula: $< f, g >= \sum_{i=1}^{s} < f, \chi_i >< \chi_i, g >$ where $\chi_1, \ldots, \chi_s$ are the irreducible characters of $G$.

4. State and prove Maschke's theorem.

5. Define a left semisimple ring. Show that any finitely generated left $A$-module over such a ring, is semisimple.

6. Prove that the matrix exponential maps a neighbourhood of $O$ in $M_n(\mathbb{R})$ homeomorphically onto a neighbourhood of $Id$ in $GL_n(\mathbb{R})$.

7. Let $A \in M_n(\mathbb{C})$. Show that the columns of the matrix $e^{tA}$ form a basis for the vector space of solutions of $\frac{dX}{dt} = AX$. Here $X$ is a column vector with $X^t = (x_1(t), \cdots, x_n(t))$.

8. If $\rho : G \to GL(V)$ is a finite-dimensional complex representation of a finite group, define the 'contragradient' representation $\rho^* = G \to GL(V^*)$ and the representation $\rho \otimes \rho : G \to GL(V \otimes V)$. Find $\chi_{\rho^*}, \chi_{\rho \otimes \rho}$.

9. Let $G \subset GL(n, \mathbf{C})$ be a finite group such that the action of $G$ on $\mathbf{C}^n$ is irreducible. Prove that the centre of $G$ must be cyclic.